Collective excitations of BEC under anharmonic trap position jittering

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Abstract. Collective excitations of a Bose-Einstein condensate under periodic oscillations of a quadratic plus quartic trap position has been studied. A coupled set of variational equations is derived for the width and the condensate wave function center. Analytical expressions for the growth of oscillation amplitudes in the resonance case are derived. It is shown that jittering of an anharmonic trap position can cause double resonance of the BEC width and the center of mass oscillation in the wide range of the BEC parameters values. The predictions of variational approach are confirmed by full numerical simulations of the 1D GP equation.

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1. Introduction

The investigation of low energy collective excitations is important for understanding of dynamics of the atomic quantum fluids (see the review [1]). Most of such theoretical and experimental studies has been performed for the condensate trapped in a harmonic (parabolic) trap. The description of the wavefunction dynamics in such a trap has many simplifying properties both for repulsive and for attractive interactions between atoms. The theory is based on the Gross-Pitaevskii equation, which is the nonlinear Schrödinger equation with the linear oscillator potential. The analysis for the repulsive condensate shows that in such potential the motion of the center of mass of condensate is decoupled from the oscillations of the condensate width. This observation is also valid for the case of attractive BEC, where in a quasi 1D geometry the matter wave soliton can exist. For the solitonic wave in a parabolic potential, the center-of-mass motion is well known to be completely decoupled from the internal excitations and it represents the analog of the Kohn theorem for solitonic wave packet [2]. It can be shown both on the level of the symmetries of 1D GP equation and using the moments method [3]. The separate resonances in the soliton width and in the position has been investigated in [4]. Resonances by the periodic variation of the scattering length has been considered for 1D Bose gas in [5].

For the case of an anharmonic trap potential the evolution of translational mode (motion of the center of mass) and the internal mode (oscillation of the width) becomes coupled. It gives rise to the possibility to control internal modes by manipulating the position of the trap. Such possibility may be useful also in creation of new technological devices, including quantum computers [6] and ultra sensitive interferometers [7]. Frequencies for low-energy excitation modes of a one-dimensional Bose-Einstein condensate with repulsive interaction between atoms in a quadratic plus quartic trap have been calculated in [8]. An approximate solution to describe the dynamics of Bose- Einstein condensates in anharmonic trapping potentials based on scaling solutions for the Thomas- Fermi radii has been presented in [9].

In this work we study effect of periodic oscillations of the anharmonic elongated trap potential position on the dynamics of BEC confined in this trap. We will consider the case when the anharmonicity and the oscillations amplitude are small.

2. The model

The dynamics of a trapped quasi-one dimensional Bose gases in elongated in the longitudinal direction anharmonic trap can be described in the framework of the 1D Gross-Pitaevskii equation

$$i\hbar\phi_t = -\frac{\hbar^2}{2m}\phi_{xx} + V(x,t)\phi + g_{1D}|\phi|^2\phi, \tag{1}$$

with the total number of atoms $N = \int |\phi|^2 dx$. This equation is obtained in the case of a highly anisotropic external potential under the assumption that the transversal trapping

potential is harmonic: $V(y,z) = m\omega_\perp^2(y^2+z^2)/2$ and $\omega_\perp \gg \omega_x$. Under such conditions we can seek the solution of a 3D equation in the form $U(x,y,z;t) = R(y,z)\phi(x,t)$ where $R_0^2 = m\omega_\perp \exp(-m\omega_\perp \rho^2/\hbar)/(\pi\hbar)$. Averaging in the radial direction (i.e. integrating over the transversal variables) we have equation (1) for the dynamics of the gas in longitudinal direction. The effective one dimensional mean field nonlinearity coefficient $g_{1D} = 2\hbar a_s \omega_\perp$, with a_s is the atomic scattering length. $a_s > 0$ corresponds to the Bose gas with a repulsive interaction between atoms and $a_s < 0$ to attractive interaction.

The dimensionless form of equation (1)

$$i\psi_t + \frac{1}{2}\psi_{xx} - V(x,t)\psi - g|\psi|^2\psi = 0, \tag{2}$$

can be obtained by setting:

$$t = \omega_x t,\ l = \sqrt{\frac{\hbar}{m\omega_x}},\ x = \frac{x}{l},\ \psi = \sqrt{2|a_s|\omega_\perp/\omega_x}\phi,$$

with $g = \pm 1$ for the repulsive and attractive two-body interactions respectively.

3. Variational analysis

To describe the collective oscillations of Bose gas in an anharmonic trap we employ the variational approach. For this purpose, we use the gaussian trial function for the wavefunction $\psi(x,t)$

$$\psi(x,t) = A(t)exp(-\frac{(x-x_0(t))^2}{2\eta^2(t)} + k(t)(x-x_0(t)) + \\ + \frac{ib(t)(x-x_0(t))^2}{2} + i\varphi(t)), \tag{3}$$

where A, η, b, x_0 and φ are the amplitude, width, chirp, center of mass and linear phase, respectively. The trap potential is chosen of the form $V(x) = V_2(x-c(t))^2 + V_4(x-c(t))^4$, where c(t) is an external parameter describing forced motion of the center of the trap.

Using this ansatz in obtaining the Euler-Lagrange equations we come to the following system of equations for the width and the center of mass of the wave packet

$$\eta_{tt} = \frac{1}{\eta^3} - 2\eta V_2 - 6V_4\eta^3 - 12V_4\eta(x_0-c)^2 + \frac{gN}{\sqrt{2\pi}\eta^2}, \tag{4}$$

$$x_{0tt} = -2V_2(x_0-c) - 6V_4\eta^2(x_0-c) - 4V_4(x_0-c)^3. \tag{5}$$

Linearizing (4) and (5) around the equilibrium points ($\eta_{tt} = 0,\ x_{0tt} = 0$) we get the following set of equations

$$\delta_{tt} = -w_\eta^2\delta - 12V_4\eta_s(x_0-c)^2, \tag{6}$$

$$x_{0tt} = -w_x^2(x_0-c), \tag{7}$$

where η_s is the equilibrium point of the width, $\delta = \eta - \eta_s$ is the deviation from the equilibrium point, w_η and w_x are determined by expressions

$$w_\eta^2 = 2V_2 + 18V_4\eta_s^2 + \frac{3}{\eta_s^4} + \frac{\sqrt{2}gN}{\sqrt{\pi}\eta_s^3} + 12V_4x_s^2,$$
$$w_x^2 = 2V_2 + 6V_4\eta_s^2 + 12V_4x_s^2. \quad (8)$$

For the excitation frequencies we have

$$w_{1,2} = \left(\frac{w_\eta^2 + w_x^2 + - \sqrt{(w_\eta^2 - w_x^2)^2 + 4k_1k_2}}{2}\right)^{\frac{1}{2}}, \quad (9)$$

where

$$k_1 = -24V_4\eta_s x_s,$$
$$k_2 = -12V_4\eta_s x_s. \quad (10)$$

Taking into account that in the equilibrium point $x_s = 0$ and $w_\eta > w_x$ we get

$$w_1 = w_\eta = \sqrt{\frac{\sqrt{2}gN}{\eta_s^3} + 2V_2 + 18V_4\eta_s^2 + \frac{3}{\eta_s^4}},$$
$$w_2 = w_x = \sqrt{2V_2 + 6V_4\eta_s^2}. \quad (11)$$

4. Resonance

Let us suppose oscillation of the trap position to be periodical, viz $c(t) = hsin(wt)$, where w is the oscillation frequency. As easily seen from the linearized equations, the center of mass and width oscillations behave like periodically driven oscillator with the "external forces" $c^2(t)$ in equation (6) and $c(t)$ in equation (7). This means that the frequency of the "external force" is equal to $2w$ in the first equation and to w in the second. Then a double resonance in oscillations of the center of mass and the width is possible when $w = w_x = w_\eta/2$.

To describe the resonance growing of the width and center of mass oscillations in equations (6) and (7) we seek δ and x_0 as $\delta = A(t)sin(2wt + \phi_1)$ and $x_0 = B(t)sin(wt + \phi_2)$. Supposing that $A(t)$ and $B(t)$ weakly depends on time and substituting these expressions into (6) and (7) and assembling coefficients of $sin(2wt + \phi 1)$, $cos(2wt + \phi 1)$, $sin(wt + \phi 2)$ and $cos(wt + \phi 2)$ we come to the following differential equations for $A(t)$ and $B(t)$

$$A_t = \frac{3V_4\eta_s}{w}B^2,$$
$$B_t = \frac{hw}{4} + \frac{12V_4\eta_s}{w}AB. \quad (12)$$

For small amplitudes $|A| << 1$ and $|B| << 1$ we have the expressions

$$A(t) = \frac{1}{16}V_4\eta_s h^2 wt^3,$$
$$B(t) = \frac{hw}{4}t, \quad (13)$$

which describe the growth of oscillation amplitudes.

5. Numerical simulations

We have carried out a series of time dependent simulations of the system evolution based on the variational approach using equations (4) and (5) as well as exact numerical computations of the full Gross-Pitaevsky (GP) equation (2). In our numerical simulations of the GP equation (2) we discretize the problem in a standard way, with the time step dt, and spatial step dx, so ψ_j^k approximates $\psi(jdx, kdt)$. More specifically we approximate the governing equation (2) with the semi-implicit Crank-Nickolson scheme using split-step method [10]. The results of numerical simulations of both PDE and ODE models are presented below. In all ODE and PDE simulations the norm of the BEC wave packet is taken to be $N = 1$.

Figure 1 depicts double resonance in oscillations of the width and center of mass of the repulsive condensate. In PDE simulations the initial wave packet is taken in the ground state. The trap center position oscillates in time periodically as $c(t) = hsin(wt)$ with the amplitude $h = 0.1$. The parameters of the trap potential are $V_2 = 0.5$ and $V_4 = 0.0005$. The value of the forced oscillation frequency $w = w_x$ at the condition $w_x = w_\eta/2$. Necessary value of the nonlinearity coefficient g providing this condition is obtained by solving a set of equations

$$\begin{aligned} &w_x = w_\eta/2, \\ &\frac{1}{\eta_s^3} - 2\eta_s V_2 - 6V_4\eta_s^3 + \frac{gN}{\sqrt{2\pi}\eta_s^2} = 0. \end{aligned} \tag{14}$$

Here the second equation determines the equilibrium point of equation (4).

As seen, unlike the harmonic case, in an anharmonic trap potential the forced oscillations of the trap center position induce oscillations not only in the condensate center of mass but also in the condensate width. In the figure for comparison full GPE and ODE simulations of the width and center of mass oscillations are shown. Theoretical prediction is shown by skirting lines described by equation (13).

Double resonance presented in figure 1 occurs under the condition $w_x = w_\eta/2$. It corresponds to the particular value of the nonlinearity coefficient $g = 0.015$ (repulsive BEC). However here we meet with a very remarkable fact that in a wide range of the nonlinearity values the ratio w_η/w_x is close to 2.

For confirmation it, in figure 2 the ratio of eigenfrequencies values of the width and mass center oscillations versus the nonlinearity coefficient, g is shown for several values of the quadratic part of potential V_2. When g ranges from -0.4 (attractive BEC) to 0.4 (repulsive BEC) the ratio changes from 2.084 to 1.964. Closeness of the ratio value w_η/w_x to 2 makes possible the existence of double resonance in a very wide range of the Bose-Einstein condensate parameters under oscillations of the anharmonic trap potential position at the frequency $w = w_x$.

To check this assertion we carried out ODE and PDE simulations when $w_\eta/w_x \neq 2$. In figure 3, double resonance in oscillations of the repulsive condensate width and center

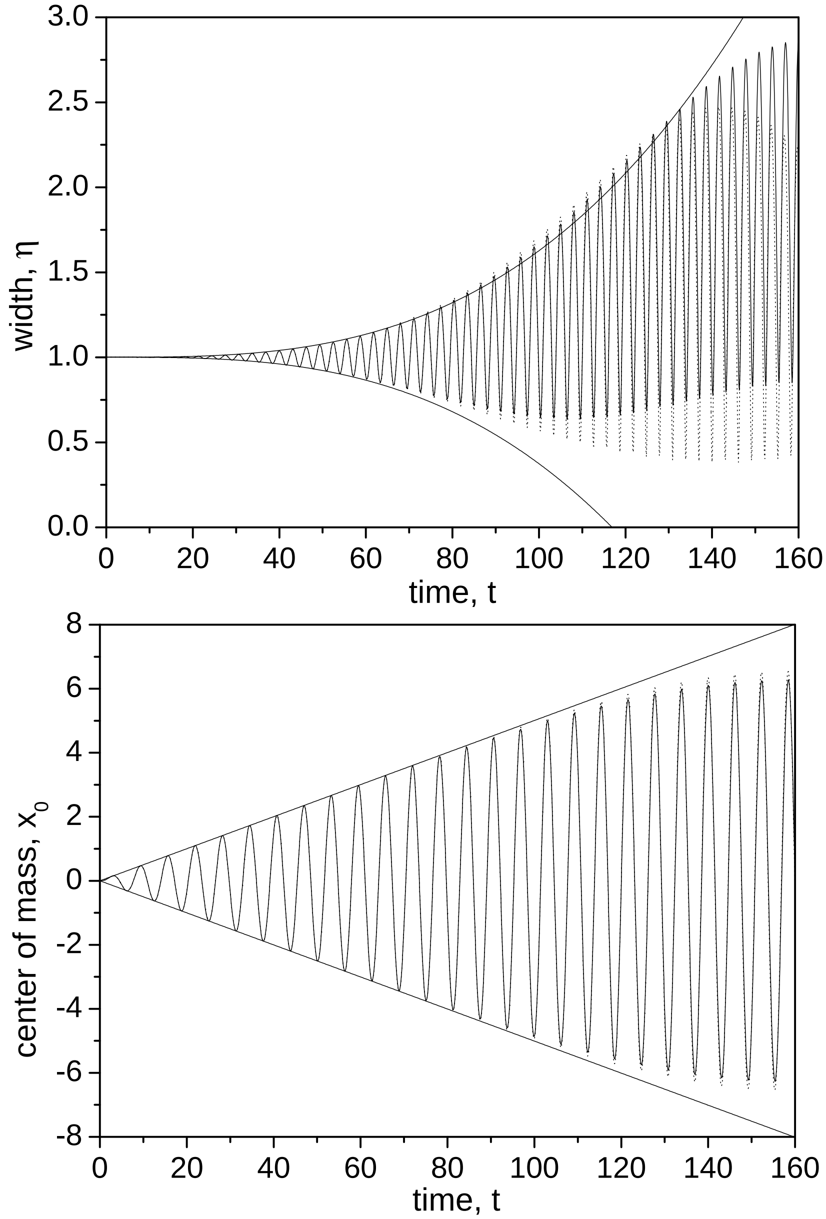

Figure 1. Resonance in oscillations of the width and the mass center of the repulsive BEC in an anharmonic trap with the forced periodical oscillation of the trap center by law $c(t) = hsin(wt)$. The parameters are $V_2 = 0.5$, $V_4 = 0.0005$, $h = 0.1$, $w_\eta/w_x = 2$ and $w = w_x$. The skirting lines present theoretical prediction of amplitude oscillations growth in the resonance. Solid and dotted lines stand for PDE and ODE simulations respectively.

of mass is presented when the nonlinearity coefficient $g = 0.4$. In this case the ratio $w_\eta/w_x = 1.966$. Here the parameters are $V_2 = 0.5$, $V_4 = 0.0005$, $h = 0.1$. The value of the forced oscillation is taken $w = w_x$. In spite of that the ratio of eigenfrequencies of the BEC width and center of mass oscillation is not equal to 2, one can observe double resonance in the oscillations.

Simulations of double resonance presented in figures 1 and 3 relate to the case of the

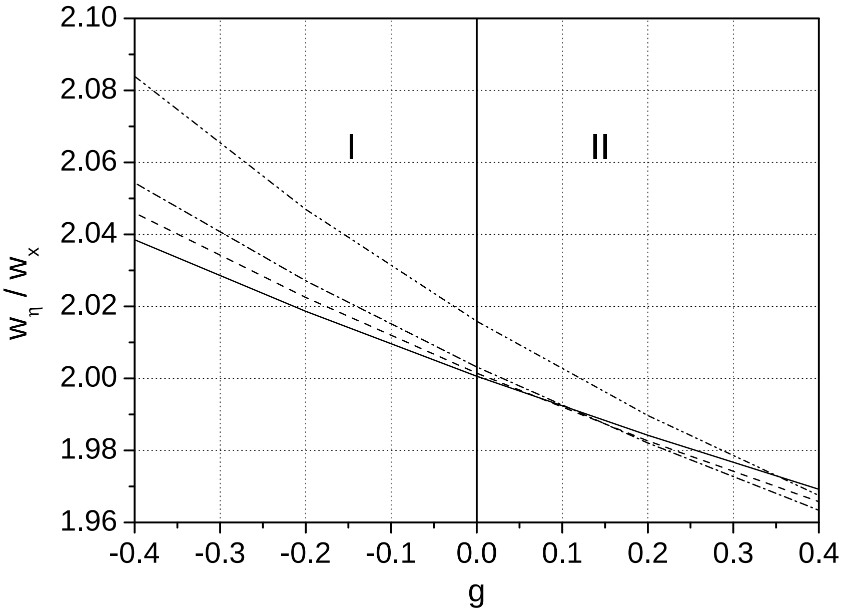

Figure 2. Ratio of eigenfrequencies of the BEC width and center of mass oscillations versus the nonlinearity g for different values of V_2. Solid, dashed, dot-dashed and dot-dot-dashed lines stand for the cases $V_2 = 0.9,\ 0.5,\ 0.3,\ 0.1$ respectively. Field I corresponds to attractive BEC and field II to repulsive one

repulsive condensate. Let us now consider an attractive condensate where the matter wave solitons can exist. As seen from figure 2 for the attractive BEC ($g < 0$) the ratio w_η/w_x is not equal to 2 and exact resonance is impossible in this case. Nevertheless the ratio w_η/w_x remains to be close to 2 (in considered range of the BEC parameters) and one can expect resonant behavior of the width and the center of mass oscillations of the attractive BEC under forced oscillations of the trap potential position with the frequency $w = w_x$.

In figure 4 resonant behavior of oscillations of the attractive condensate width and center of mass is depicted when the nonlinearity coefficient $g = -0.4$. In this case the ratio $w_\eta/w_x = 2.046$. Here the parameters are $V_2 = 0.5$, $V_4 = 0.0005$, $h = 0.1$. The value of the forced oscillation is taken $w = w_x$. As in the case of *repulsive* BEC one can observe that the oscillations close to double resonance in the case of *attractive* BEC.

As seen the results of ODE simulations of the double resonance are in a good agreement with full PDE ones at times $t < 100$ and then begins to differ at larger times. At these times the amplitude of oscillations of the BEC center of mass x_0 is great and anharmonic part of the trap potential becomes noticeable that leads to difference between ODE and PDE simulations.

6. Conclusion

In this paper we have studied collective oscillations of a quasi- one- dimensional Bose gas in an ahharmonic trap under periodic oscillations of the trap position in time. To describe evolution of oscillations we use variational approach with Gausiian ansatz.

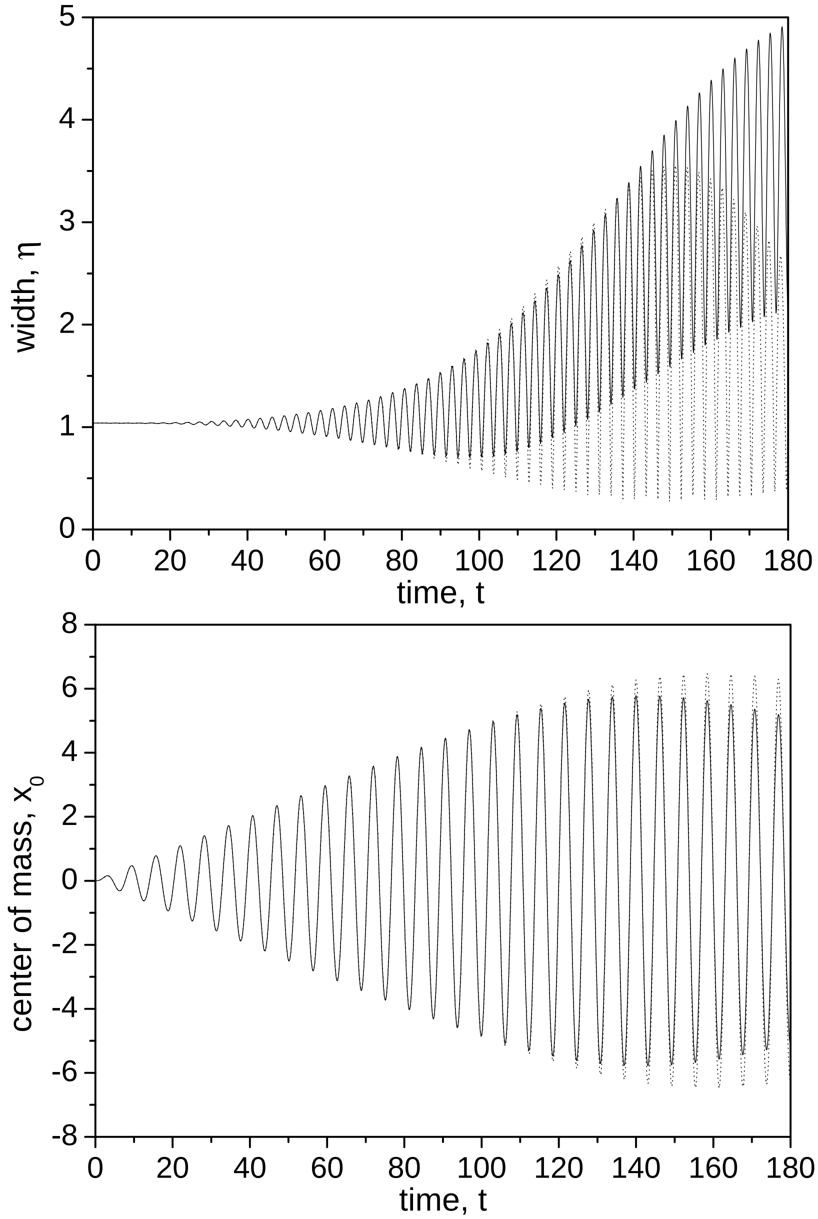

Figure 3. The width and center of mass oscillations of the repulsive BEC in an anharmonic trap with the forced periodical oscillation of the trap center with the frequency $w = w_x$ and $w_\eta/w_x = 1.966$. Nonlinear coefficient, g equals to 0.4. The parameters are $V_2 = 0.5$, $V_4 = 0.0005$, $h = 0.1$. Solid and dotted lines stand for PDE and ODE simulations respectively.

Double resonance in the condensate oscillations has been studied. Analytical expressions have been derived for the growth of oscillation amplitudes in the resonance.

Analysis of the variational equations has shown existence of a double resonance in oscillations of the center of mass and the width under forced oscillations of the trap center position, provided that the ratio of the eigenfrequencies $w_\eta/w_x = 2$ and the forced trap position oscillation frequency $w = w_x$. It is shown that for a wide range of values of the BEC parameters the ratio w_η/w_x is close to 2 and the behavior of the oscillations

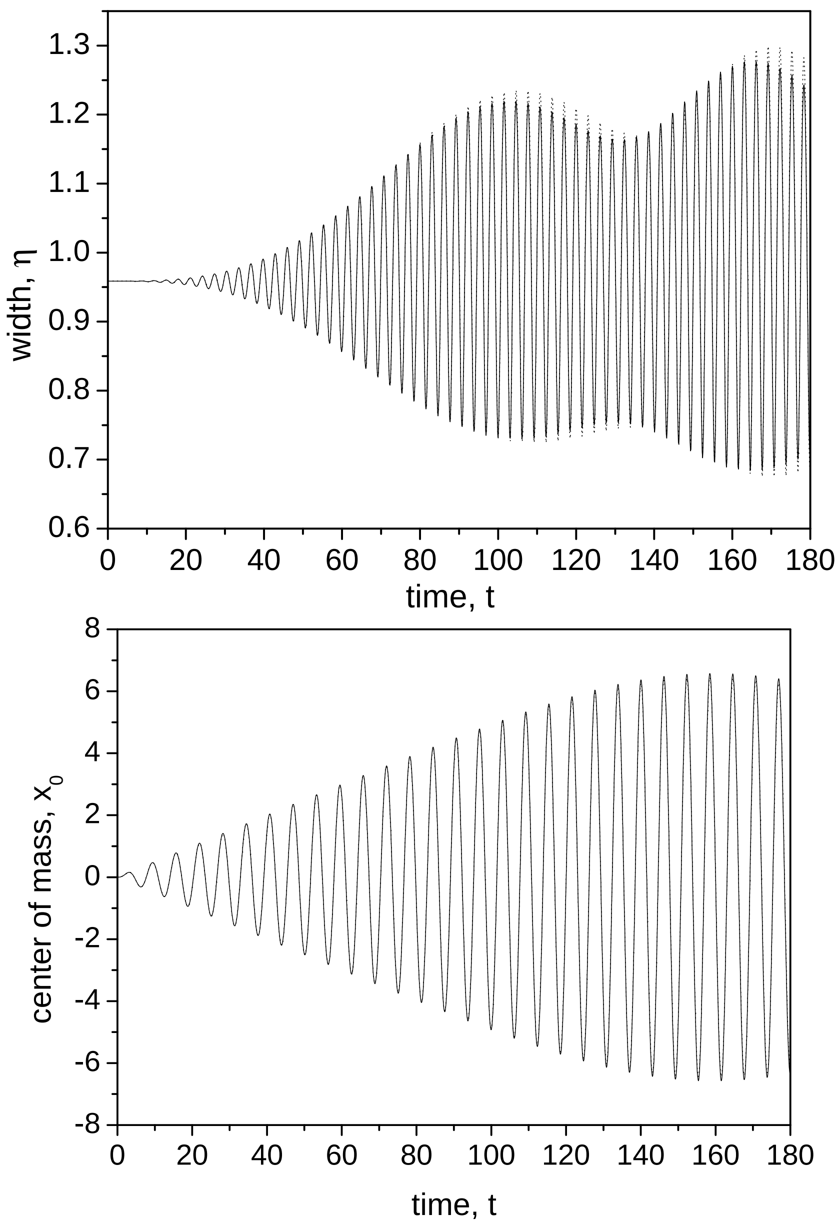

Figure 4. The width and center of mass oscillations of the attractive BEC in an anharmonic trap with the forced periodical oscillation of the trap center with the frequency $w = w_x$ and $w_\eta/w_x = 2.046$. Nonlinear coefficient, g equals to -0.4. The parameters are $V_2 = 0.5$, $V_4 = 0.0005$, $h = 0.1$. Solid and dotted lines stand for ODE and PDE simulations respectively.

is close to resonant both for repulsive and attractive Bose gases.

Theoretical predictions are confirmed by full numerical simulations of the 1D GP equation.

Acknowledgments

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